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Generalized charged static dust spheres in relativity

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Abstract. A static, bounded, spherically symmetric distribution of dust is considered; the constituents of this dust are simultaneously sources of gravitational, electrostatic and long-range scalar fields. Two kinds of scalar fields are considered—attractive and repulsive. Two classes of exact external solutions corresponding to these two types of fields are obtained; these solutions are asymptotically flat and satisfy the continuity conditions at the surface of the distribution. The internal solutions depend on two somewhat arbitrary functions of the radial coordinate. The solutions tend straightforwardly to all less general solutions found in the literature.

1. Introduction

Spherically symmetric distributions of electrically charged dust, in static condition, were studied by Bonnor (1960); he verified that equilibrium could only be maintained when the ratio of the density of charge to the density of matter remained constant: $\sigma = \pm\rho$ in his units. Later De and Raychaudhuri (1968) showed that the relation $\sigma = \pm\rho$ is a consequence of the Einstein–Maxwell equations irrespective of any symmetry, provided there is no singularity in the distribution. Very recently Wolk *et al* (1975) studied a distribution of incoherent dust, the constituents of which were supposed to be the sources of gravitational as well as of repulsive long-range scalar fields. Using an analysis similar to that of De and Raychaudhuri they found that in static systems free of singularity one should also have an identical relation between the densities of scalar charge and of matter.

We now generalize all these results by considering a static distribution of incoherent dust, charged both in the electric and in the scalar sense. Our scalar field can be either of an attractive or of a repulsive kind; as in Teixeira *et al* (1976) we call a scalar field attractive (repulsive) when it produces attraction (repulsion) between scalar charges of the same sign, unlike (like) what happens in electrostatics. For definiteness we have considered a distribution with spherical symmetry, and obtained the external and internal solutions.

2. Basic equations

In the Einstein equations (Anderson 1967)‡

$$R^{\mu}_{\nu} = -8\pi(T^{\mu}_{\nu} - \delta^{\mu}_{\nu}T/2) \tag{1}$$

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‡ We use the notations and conventions of this book.

we take as the energy–momentum density of our system:

$$T^{\mu}_{\nu}=\rho u^{\mu}u_{\nu}+E^{\mu}_{\nu}+K^{\mu}_{\nu}; \tag{2}$$

here ρ is the mass density of a distribution with velocity u^{μ}, and E^{μ}_{ν} and K^{μ}_{ν} are the energy–momentum densities of an electromagnetic and of a long-range scalar field.

The tensor E^{μ}_{ν} is given by

$$4\pi E^{\mu}_{\nu}=F^{\mu}_{\alpha}F^{\alpha}_{\nu}-\delta^{\mu}_{\nu}F^{\alpha}_{\beta}F^{\beta}_{\alpha}/4, \tag{3}$$

where

$$F_{\mu\nu}=A_{\nu;\mu}-A_{\mu;\nu} \tag{4}$$

is the electromagnetic field, which satisfies Maxwell's equations

$$F^{\mu\nu}{}_{;\mu}=4\pi\sigma u^{\nu}, \tag{5}$$

σ being an electric charge density; subscripted and superscripted semicolons mean covariant derivatives.

The tensor K^{μ}_{ν} is given (Teixeira *et al* 1976) by

$$4\pi\gamma K^{\mu}_{\nu}=S^{;\mu}S_{;\nu}-\delta^{\mu}_{\nu}S^{;\alpha}S_{;\alpha}/2, \tag{6}$$

where $\gamma=+1$ when the scalar field S is attractive, and $\gamma=-1$ for repulsive scalar fields. Both kinds of scalar fields satisfy

$$S^{;\mu}_{;\mu}=-4\pi\gamma s, \tag{7}$$

where s is the density of the source of S.

The contracted Bianchi identities

$$2R^{\mu}_{\nu;\mu}\equiv R_{;\nu} \tag{8}$$

give for our system

$$\rho u_{\mu;\nu}u^{\nu}=\sigma F_{\mu\nu}u^{\nu}+sS_{;\mu}. \tag{9}$$

In this work we shall be concerned with spherically symmetric static systems; for such symmetry we use the line element

$$ds^2=e^{2\eta}(dx^0)^2-e^{2\alpha}\,dr^2-r^2\,e^{\alpha-\eta}(d\theta^2+\sin^2\theta\,d\phi^2), \tag{10}$$

with η and α functions of r alone; then the components of the Ricci tensor are

$$R^0_0=-(\eta_{11}+2\eta_1/r)\,e^{-2\alpha}, \tag{11}$$

$$R^1_1=-(\alpha_{11}-\alpha_1^2/2-\alpha_1\eta_1+3\eta_1^2/2-2\eta_1/r)\,e^{-2\alpha}, \tag{12}$$

$$R^2_2=R^3_3=(\eta_{11}/2-\alpha_{11}/2+\eta_1/r-\alpha_1/r-r^{-2})\,e^{-2\alpha}+r^{-2}\,e^{\eta-\alpha}, \tag{13}$$

where a subscript 1 means d/dr. All sources ρ, σ and s are functions of r alone, the same happening to the fields A_{μ} and S, and to the velocity u^{μ},

$$A_{\mu}=\delta^0_{\mu}\Phi(r), \tag{14}$$

$$u^{\mu}=\delta^{\mu}_0\,e^{-\eta}. \tag{15}$$

The electromagnetic and scalar energy momentum tensors become

$$8\pi E^{\mu}_{\nu}=\Phi_1^2\,e^{-2(\eta+\alpha)}\,\mathrm{diag}(+,+,-,-), \tag{16}$$

$$8\pi K^{\mu}_{\nu}=\gamma S_1^2\,e^{-2\alpha}\,\mathrm{diag}(+,-,+,+); \tag{17}$$

and the electrostatic and scalar field equations are now

$$(r^2 e^{-2\eta}\Phi_1)_1 = 4\pi\sigma r^2 e^{2\alpha-\eta}, \tag{18}$$

$$(r^2 S_1)_1 = 4\pi\gamma s r^2 e^{2\alpha}, \tag{19}$$

with the Bianchi identity

$$\rho\eta_1 + \sigma e^{-\eta}\Phi_1 + sS_1 = 0. \tag{20}$$

3. General exterior solutions

We put $\rho = \sigma = s = 0$ in the equations of § 2 and obtain

$$\eta_{11} + 2\eta_1/r = e^{-2\eta}\Phi_1^2, \tag{21}$$

$$\alpha_{11} + 3\eta_1^2/2 - \alpha_1^2/2 - \eta_1\alpha_1 - 2\eta_1/r = e^{-2\eta}\Phi_1^2 - 2\gamma S_1^2, \tag{22}$$

$$(\eta_{11} - \alpha_{11})/2 + (\eta_1 - \alpha_1)/r + (e^{\eta+\alpha} - 1)/r^2 = e^{-2\eta}\Phi_1^2, \tag{23}$$

$$(r^2 e^{-2\eta}\Phi_1)_1 = 0. \tag{24}$$

From the last equation we get immediately

$$\Phi_1 = -qr^{-2} e^{2\eta}, \tag{25}$$

where q is a constant of integration.

The substitution of (25) into (21) gives for η the solution

$$e^{-\eta} = \cosh(d + c/r) + (1 + q^2/c^2)^{1/2} \sinh(d + c/r), \tag{26}$$

where c and d are constants of integration; we impose $\eta = 0$ at infinity, which implies that $d = 0$.

The subtraction of (21) from (23) yields the solution

$$e^{\eta+\alpha} = (f/r)^2 \sinh^{-2}(g + f/r), \tag{27}$$

with f and g constants of integration; the imposition of $\eta + \alpha = 0$ at infinity demands that $g = 0$.

Finally (22) gives for S_1^2 the solution

$$S_1^2 = \gamma(f^2 - c^2)/r^4; \tag{28}$$

this solution is compatible with (19) which in our exterior region is expressed by $(r^2 S_1)_1 = 0$.

So the exterior solution of our problem, which tends to flatness at infinity is (subscript e for exterior)

$$g_{00} = \exp(2\eta_e) = [\cosh c/r + (1 + q^2/c^2)^{1/2} \sinh c/r]^{-2}, \tag{29}$$

$$g_{rr} = -\exp(2\alpha_e) = -(f/r)^4 \sinh^{-4}(f/r) \exp(-2\eta_e), \tag{30}$$

$$g_{\theta\theta} = -r^2 \exp(\alpha_e - \eta_e), \tag{31}$$

$$\Phi_1 = -qr^{-2} \exp(2\eta_e), \tag{32}$$

$$S_1^2 = \gamma(f^2 - c^2)/r^4, \tag{33}$$

with q, c and f constants to be associated somehow with the gravitational, electric and scalar charges. While the constant q must be real in order to have the usual physical meaning in (32), the constants c and f can be real or imaginary, independently. This spherically symmetric exterior solution ((29)–(33)) is consistent with more general results already obtained by Teixeira *et al* (1976).

The familiar Reissner–Nordström exterior solution:

$$g_{00} = 1 - 2m/r' + q^2/r'^2, \qquad g_{r'r'} = -g_{00}^{-1}, \qquad g_{\theta\theta} = -r'^2, \qquad \Phi = q/r'$$

is obtained by putting $c = f$ and then performing the radial coordinate transformation $r' = m + f \coth f/r$ with $m^2 = q^2 + f^2$; and Yilmaz's (1958) one-parameter repulsive scalar field solution

$$ds^2 = e^{-2c/r}(dx^0)^2 - e^{2c/r}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2), \qquad S_1^2 = c^2/r^4$$

is obtained by putting $q = f = 0$.

4. Interior solutions

We consider now a static, spherically symmetric distribution of incoherent dust, charged both electrostatically and scalarly (long-range, either attractive $\gamma = +1$ or repulsive $\gamma = -1$); we assume that all distributions $\rho(r)$, $\sigma(r)$ and $s(r)$ are regular. Our set of equations is now

$$\eta_{11} + 2\eta_1/r = 4\pi\rho\, e^{2\alpha} + e^{-2\eta}\Phi_1^2, \tag{34}$$

$$\alpha_{11} + 3\eta_1^2/2 - \alpha_1^2/2 - \eta_1\alpha_1 - 2\eta_1/r = -4\pi\rho\, e^{2\alpha} + e^{-2\eta}\Phi_1^2 - 2\gamma S_1^2, \tag{35}$$

$$(\eta_{11} - \alpha_{11})/2 + (\eta_1 - \alpha_1)/r + (e^{\eta+\alpha} - 1)/r^2 = 4\pi\rho\, e^{2\alpha} + e^{-2\eta}\Phi_1^2, \tag{36}$$

$$(r^2 e^{-2\eta}\Phi_1)_1 = -4\pi\sigma r^2 e^{2\alpha-\eta}, \tag{37}$$

$$(r^2 S_1)_1 = 4\pi\gamma s r^2 e^{2\alpha}, \tag{38}$$

with the contracted Bianchi identity

$$\rho\eta_1 + \sigma\, e^{-\eta}\Phi_1 + sS_1 = 0. \tag{39}$$

The subtraction of (34) from (36) again gives the solution (27); however, in our interior system we can only have regularity at the origin when both constants f and g vanish in such a way that

$$\eta_i + \alpha_i = 0, \tag{40}$$

where the subscript i stands for internal.

We next add (34) and (35), consider (40) and get the quadratic first-order relation

$$\eta_1^2 - e^{-2\eta}\Phi_1^2 + \gamma S_1^2 = 0; \tag{41}$$

from the system (39) and (41) we get

$$\Phi_1 = -Q\, e^{\eta}\eta_1, \tag{42}$$

$$S_1^2 = \gamma(Q^2 - 1)\eta_1^2, \tag{43}$$

where the function $Q(r)$ is a combination of ρ, σ and s given by

$$(Q^2 - 1)s^2 - \gamma(\rho - Q\sigma)^2 = 0. \tag{44}$$

The interior solution of our system can then be specified by the two functions $\eta(r)$ and $Q(r)$; having chosen these two functions we get Φ_1 and S_1 from (42) and (43), respectively; next we obtain

$$4\pi\rho = (\eta_{11} + 2\eta_1/r)\,e^{2\eta} - \Phi_1^2, \tag{45}$$

$$4\pi\sigma = -r^{-2}\,e^{3\eta}(r^2\,e^{-2\eta}\Phi_1)_1, \tag{46}$$

$$4\pi s = \gamma r^{-2}\,e^{2\eta}(r^2 S_1)_1. \tag{47}$$

We see from (43) that $Q^2(r)$ must be everywhere greater (less) than unity, for attractive (repulsive) scalar fields, and we see that the choice of $\eta(r)$ and $Q(r)$ are not completely arbitrary, since the density of mass $\rho(r)$ in (45) must be positive. Some small additional restrictions on η and Q will arise from the continuity conditions on the boundary of the sphere.

Bonnor's (1960) well known electrostatic solutions correspond to the particular case $Q^2 = 1$; his radial coordinate r' is related to our r by $r' = r\exp(-\eta)$. Another particular case is that of vanishing density of electric charge, a situation which can be represented by $Q = 0$; then (43) demands that $\gamma = -1$. Indeed, in the absence of electrostatic repulsion a repulsive scalar field is required for balancing the gravitational attraction; this particular case was studied by Yilmaz (1958).

5. Complete solutions

In matching the regular interior solutions with the asymptotically flat exterior solutions we impose the continuity of the fields g_{00}, g_{rr}, Φ, S, and of the radial derivatives dg_{00}/dr, $d\Phi/dr$, dS/dr on the boundary r_0 of the sphere.

The continuity of g_{00} and g_{rr} implies, from (29), (30) and (40), that $f = 0$, or equivalently $\alpha_e = -\eta_e$. Then from (33) we see that for attractive scalar fields ($\gamma = +1$) the parameter c must be imaginary, $c = ib$ (b real), and from (29) we note that we must have $b^2 < q^2$ for $\gamma = +1$. And we see that for repulsive scalar fields ($\gamma = -1$) the parameter c is real. So if we define a positive constant m according to

$$m^2 = q^2 - \gamma b^2 \tag{48}$$

we have two different expressions for the exterior g_{00} metric coefficient:

$$g_{00} = \exp(2\eta_e) = \begin{cases} [\cos b/r + (m/b)\sin b/r]^{-2} & (\gamma = +1), \quad (49) \\ [\cosh b/r + (m/b)\sinh b/r]^{-2} & (\gamma = -1); \quad (50) \end{cases}$$

in both cases $\gamma = \pm 1$ we have the exterior quantities

$$g_{rr} = r^{-2} g_{\theta\theta} = -\exp(-2\eta_e), \tag{51}$$

$$\Phi_{1e} = -qr^{-2}\exp(2\eta_e), \tag{52}$$

$$S_{1e} = \gamma b/r^2. \tag{53}$$

The continuity of g_{00} and dg_{00}/dr implies that the internal $\eta_i(r)$ used in (42)–(47) must satisfy

$$\eta_i(r_0) = \eta_e(r_0), \qquad \eta_{1i}(r_0) = \eta_{1e}(r_0), \tag{54}$$

where $\eta_e(r)$ is that given in (49) or (50). Finally the continuity of $\mathrm{d}\Phi/\mathrm{d}r$ and of $\mathrm{d}S/\mathrm{d}r$ on the boundary r_0 both give the same restriction, say from (42) and (52):

$$Q(r_0) = q(r_0^2\eta_{1e}(r_0))^{-1}\exp(\eta_e(r_0)). \tag{55}$$

6. Discussion

One sees from the asymptotic expressions of (49)–(53) that the parameters m, q and b represent, in the weak-field approximation, the mass, the electric charge and the scalar charge of the sphere, respectively. While in the absence of scalar charges one observes that $q^2 = m^2$ (with a relation between the corresponding densities, $\sigma^2 = \rho^2$), and in the absence of electric charge one has $b^2 = m^2$ (with the relation $s^2 = \rho^2$), in our general case one has $m^2 - q^2 + \gamma b^2 = 0$, but the relation (44) between the corresponding densities involves a somewhat arbitrary function $Q(r)$.

Also in connection with the relation $m^2 = q^2 - \gamma b^2$ one should remark that for attractive scalar fields ($\gamma = +1$) the parameter m can only be interpreted as a mass parameter when $b^2 < q^2$; a classical picture to see the origin of this result has already been tried by Teixeira *et al* (1976).

In the case of vanishing total scalar source ($b \to 0$) the two external metric coefficients (49) and (50) approach each other, giving as a limiting case

$$\mathrm{d}s^2 = (1 + m/r)^{-2}(\mathrm{d}x^0)^2 - (1 + m/r)^2[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)];$$

this result has already been obtained by Papapetrou (1947), his radial coordinate r' being related to our r by $r' = r + m$.

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